Quantum resource theories of quantum channels

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Overview

- Brief intro of quantum resource theories
- From states to channels
- What is the power/cost of a quantum channel?
  - From different resource perspectives?
  - Under different settings?
- Application of resource theory to quantum channel distinguishability
- Summary and outlook

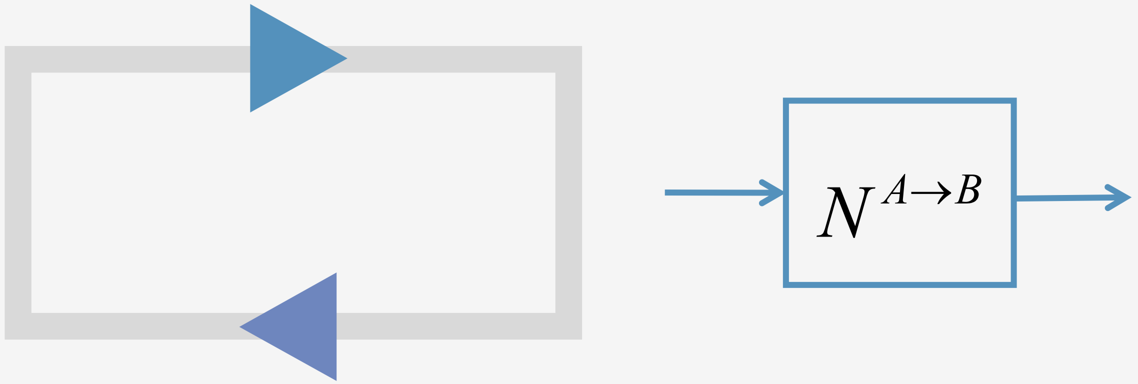
What are resources?

- Information
- Energy
- Entanglement
- Coherence
- …
- **Quantum channel**

- Resources can be converted under certain conditions.
- We need a framework and a development kit to study these quantum resources.

Our focus

Resources



What is the power/cost of quantum channels from the resource perspective?

What are quantum resource theories?

- A QRT models what we can physically accomplish given constraints on physical operations.

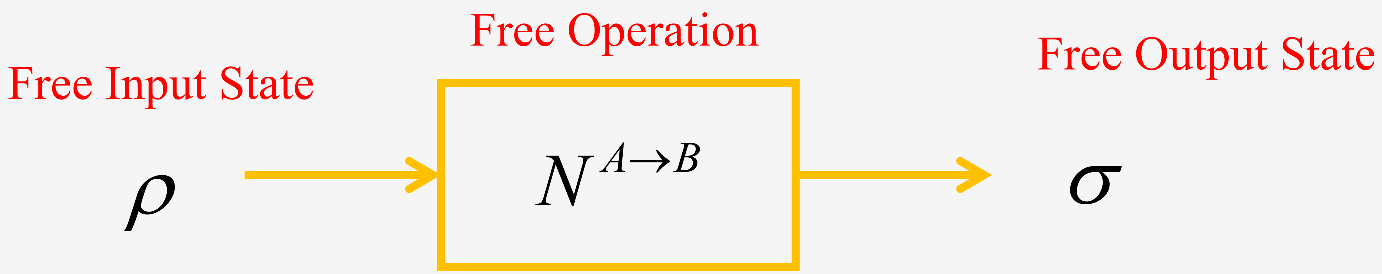
Quantum resource theories

Eric Chitambar and Gilad Gour
Rev. Mod. Phys. **91**, 025001 – Published 4 April 2019

- Resource theories offer a systematic and powerful framework for studying the power and limits of quantum resources.
- Resource theories for static resources (entanglement, coherence,, randomness, magic, thermodynamics) and dynamic resources (communication), etc.

Framework of quantum resource theories

- Free states
- Free operations
- The Golden Rule

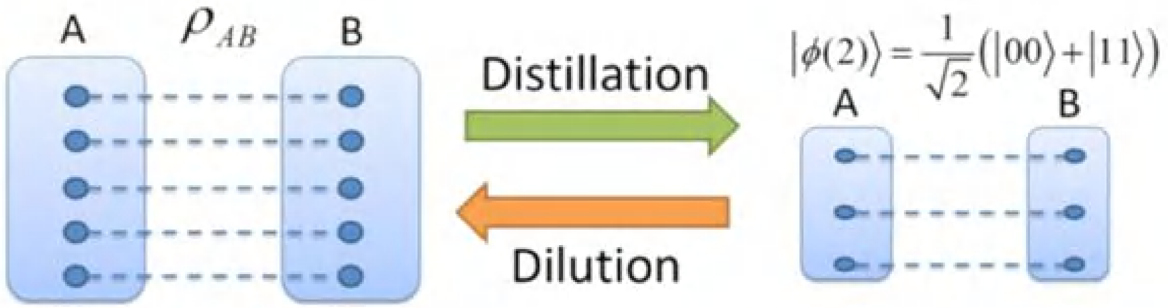


- Detection, quantification, manipulation and applications of quantum resources.
- Many resource-theoretic Tasks.

Warm-up example – entanglement theory

- Entanglement is a property of a composite physical system that cannot be generated by local operations and classical communication (LOCC).
- Resource theory of entanglement: separable states + LOCC.
- Golden units - maximally entangled states

- Transformation between resource states
- Distillable entanglement
- Entanglement cost



More Examples

	Entanglement	Magic	Coherence
Free states	Separable states	Stabilizer states	Incoherent state
Free operations	LOCC operations	Stabilizer operations	Incoherent operations
Key task	Entanglement distillation	Distilling magic state (e.g. T state)	Coherence distillation

From quantum states to quantum channels

- Resource theory naturally goes to higher order, but with motivations:
  - Channels are resources (e.g., Shannon theory).
  - Quantum channels can represent dynamical resources.
- More complicated but also more fruitful structure.

- Recent progress on resource theory of channels, see, e.g.,
  - Liu, Winter (1904.04201); Liu, Yuan (1904.02680); Gour (1808.02607); Li, Bu, Liu (1812.02572); Gour, Wilde (1808.06980); Faist, Berta, Brandão (1807.05610)
  - XW, Wilde (1809.09592); Fang, XW, Tomamihcel, Berta (1807.05354); XW, Wilde, Su (1903.04483); XW, Wilde (1907.06306);

What is a quantum channel?

- Quantum Channel or quantum process: completely positive (CP) trace-preserving (TP) linear map N.

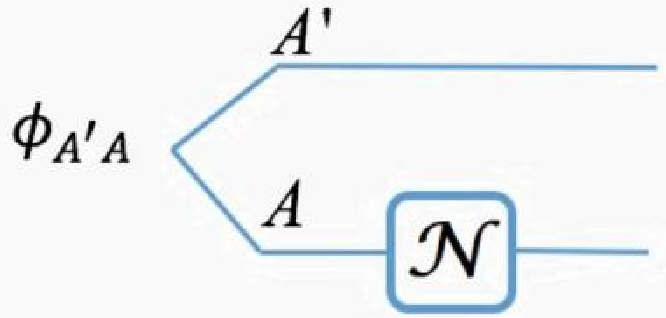
- Choi-Kraus representation

$$\mathcal{N}(\cdot) = \sum_k E_k \cdot E_k^\dagger, \text{ with} \sum_k E_k^\dagger E_k = \mathbb{1}.$$

- Stinespring rep. $\mathcal{N}(\cdot) = \mathrm{Tr}_E(V \cdot V^\dagger)$,
  with isometry $V(A \to BE)$.
- Choi-Jamiołkowski representation

$$J_{\mathcal{N}} = (\mathrm{id}_{A'} \otimes \mathcal{N})|\Phi_{A'A}\rangle\langle\Phi_{A'A}|,$$

with $|\Phi_{A'A}\rangle = \sum_k |k_{A'}\rangle|k_A\rangle$.

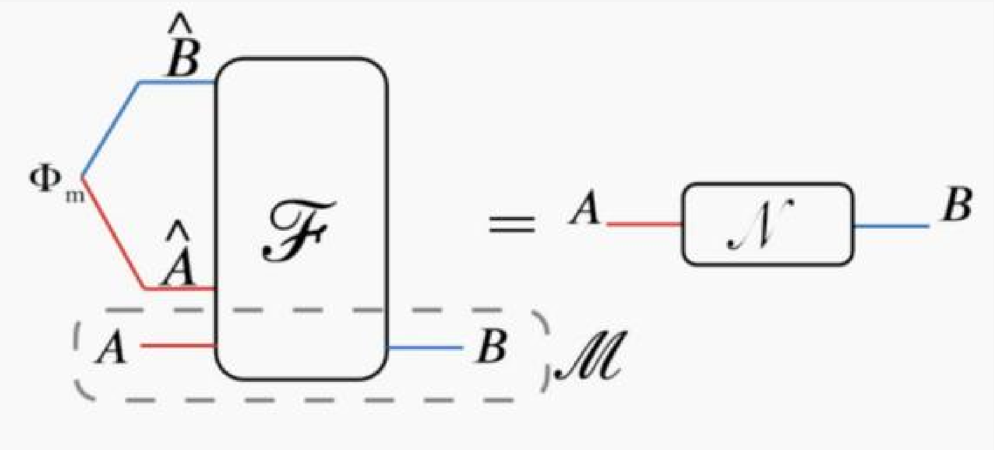


Note that $J_{\mathcal{N}} \geq 0, \mathrm{Tr}_B J_{\mathcal{N}} = \mathbb{1}$.

What is the role of quantum channels in QRT?

- Free quantum operations.
- The quantum channel itself is a kind of quantum resource.
- What is the fundamental quantum cost of implement thequantum channels?
- How many quantum resources can be generated from the quantum channels?

What is the cost to realize a quantum channel?



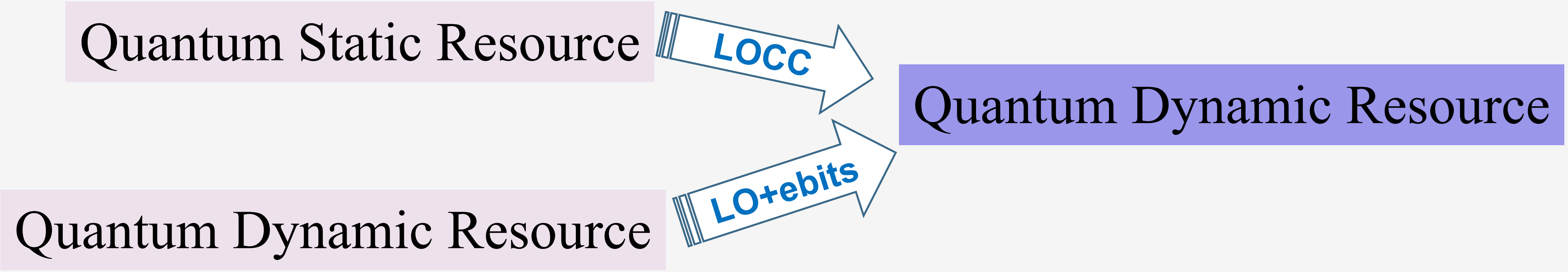
Resource trade-off

- classical communication (in protocol $\mathcal{F}$)
- shared entanglement Φ

- A good example to start with is quantum teleportation (Bennett et al.'93). In this protocol, one needs two classical bits and one ebit to realize a noiseless qubit channel.
- When classical communication is free, what is the entanglement cost?
- When entanglement is free, what is the communication cost?

What is the cost to realize a quantum channel?

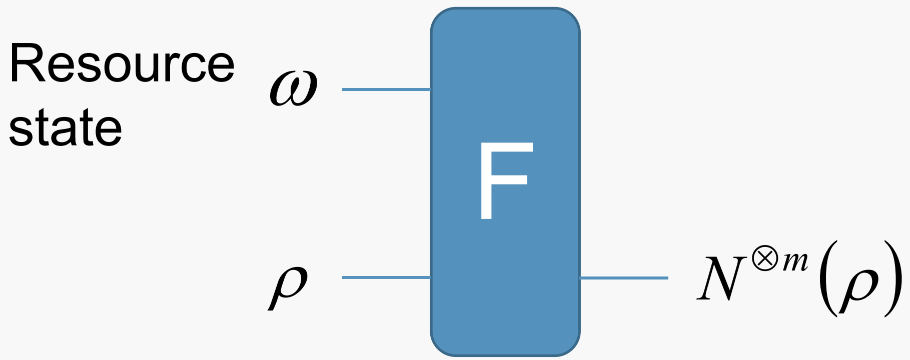
For the entanglement theory of quantum channels

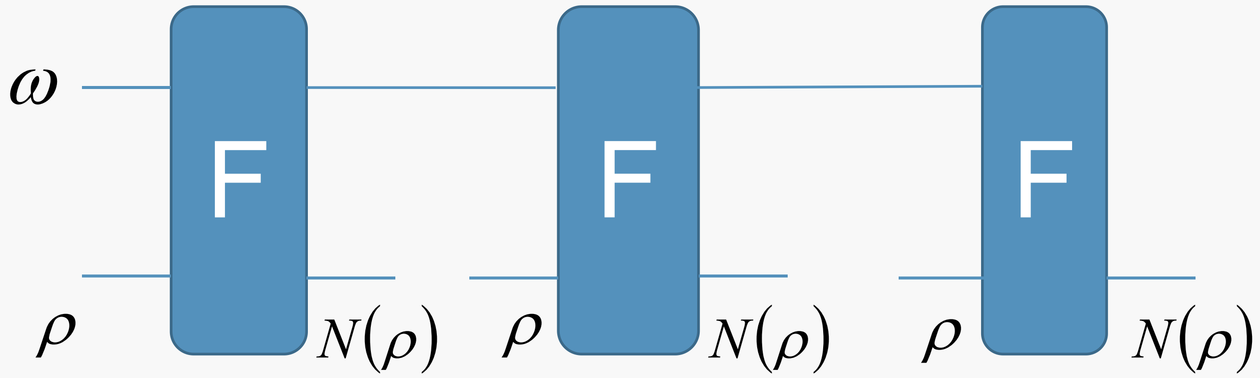


- Static resource cost under free operations (e.g., entanglement cost of a channel)
- Dynamic resource cost under free operations (the most famous example is quantum Shannon theory).

Protocols - adaptive vs parallel

- For the RT of quantum channels, we are interested in both parallel and adaptive regimes.
- The main idea behind sequential channel simulation is to simulate m uses of the channel N in such a way that they can be called in an arbitrary order, i.e., on demand when they are needed.



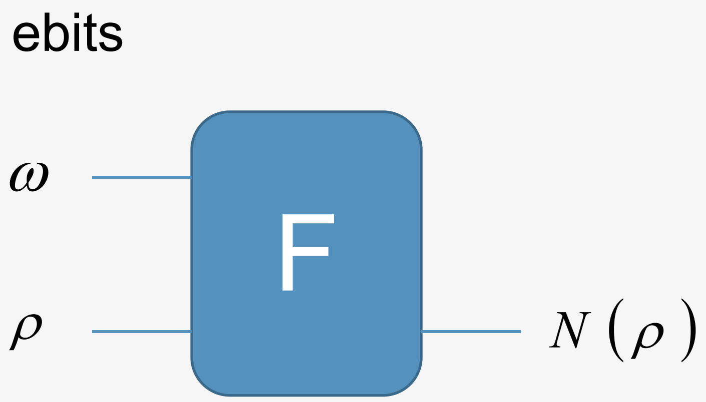


Reuse the resource state in an adaptive way

- Sequential channel simulation is stronger than parallel simulation, thus has a higher resource cost.
- Compatible with a discrimination strategy that can test the the above simulation in a sequential way (Chiribella et al'09; Gutoski'12).

Resource theory of entanglement for quantum channels

What is the entanglement cost of a quantum channel?



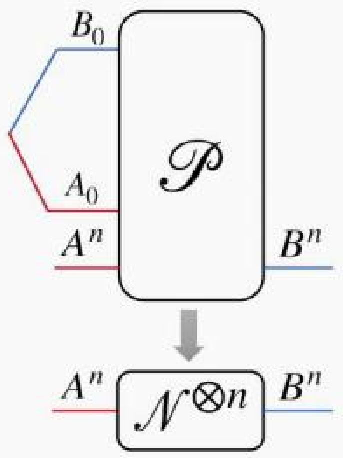
- When classical communication is free, Berta, Brandao, Christandl, Wehner'11 introduced the entanglement cost of a quantum channel.
- It is the minimal rate at which entanglement (between sender and receiver) is needed in order to simulate many copies of a quantum channel in the presence of free classical communication.

$$E_C(\mathcal{N}) := \inf\{\log r : \lim_{n\to\infty} \inf_{\mathcal{F}\in\mathrm{LOCC}} \|\mathcal{N}^{\otimes n} - \mathcal{F}(\cdot \otimes \Phi(2^{rn}))\|_\diamond = 0\}$$

And they proved that

$$E_C(\mathcal{N}) = \lim_{n\to\infty} \max_{\psi^n} E_F(\mathcal{N}^{\otimes n} \otimes \mathcal{I}(\psi^n))/n.$$

Exact entanglement cost of quantum channel



- **One-shot exact entanglement cost** of $\mathcal{N}_{A\to B}$, under Ω operations

$$E_\Omega^{(1)}(\mathcal{N}) = \inf_{\Lambda\in\Omega}\left\{\log d : \mathcal{N} = \Lambda_{A_0B_0A\to B}(\cdot \otimes \Phi^d_{A_0B_0})\right\}.$$

- **Exact parallel entanglement cost** of $\mathcal{N}_{A\to B}$ under Ω operations

$$E_\Omega^{(p)}(\mathcal{N}) = \liminf_{n\to\infty}\frac{1}{n}E_\Omega^{(1)}(\mathcal{N}^{\otimes n}).$$

- When LOCC is free, the problem is extremely difficult. For the mixed states, it is unsolved.
- We thus consider a larger set of free operations called PPT operations,

Partial transpose: $|ij\rangle\langle kl|_{AB}^{T_B} = |il\rangle\langle kj|_{AB}$.

Positive-partial transpose (PPT)

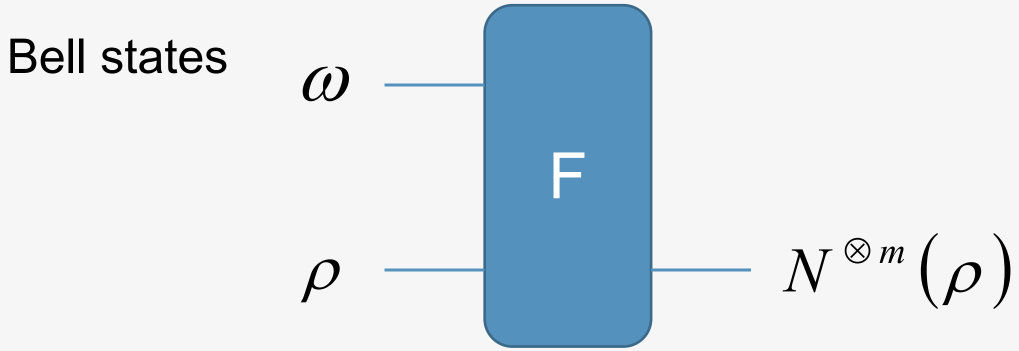
$$\rho^{T_B} \geq 0 \quad \text{(Peres–Horodecki criterion for separability).}$$

The most common set of quantum operations beyond LOCC consists of **PPT operations** ($T_{B'} \circ \Lambda_{AB\to A'B'} \circ T_B$ is also completely positive).

Exact entanglement cost of quantum channel

- When PPT operations are free, we obtain

$$E^{(1)}_{\mathrm{PPT}}(\mathcal{N}) = \inf \log m$$
$$\text{s.t.} \; -(m-1)\, Q_{AB}^{T_B} \le (J_{AB}^{\mathcal{N}})^{T_B} \le (m+1)\, Q_{AB}^{T_B},$$
$$Q_{AB} \ge 0, \;\; \mathrm{Tr}_B\, Q_{AB} = \mathbb{1}_A$$



- What is the asymptotic exact parallel entanglement cost?

$$E^{(p)}_{\mathrm{PPT}}(\mathcal{N}) = \liminf_{n\to\infty} \frac{1}{n} E^{(1)}_{\mathrm{PPT}}\left(\mathcal{N}^{\otimes n}\right)$$

Exact entanglement cost of quantum channel

κ-entanglement of a quantum channel

We define the κ-entanglement of a quantum channel $\mathcal{N}_{A\to B}$ as

$$E_\kappa(\mathcal{N}) = \inf\{\log \|\mathrm{Tr}_B\, Q_{AB}\|_\infty : -Q_{AB}^{T_B} \le (J_{AB}^{\mathcal{N}})^{T_B} \le Q_{AB}^{T_B},\ Q_{AB} \ge 0\},$$

where $J_{AB}^{\mathcal{N}}$ is the Choi operator of $\mathcal{N}$.

- Introduce the one-shot SDP sandwiched approximation

$$\log\left(2^{E_\kappa(\mathcal{N})} - 1\right) \le E_{\mathrm{PPT}}^{(1)}(\mathcal{N}_{A\to B}) \le \log\left(2^{E_\kappa(\mathcal{N})} + 2\right)$$

- Apply the SDP duality theory to get the additivity

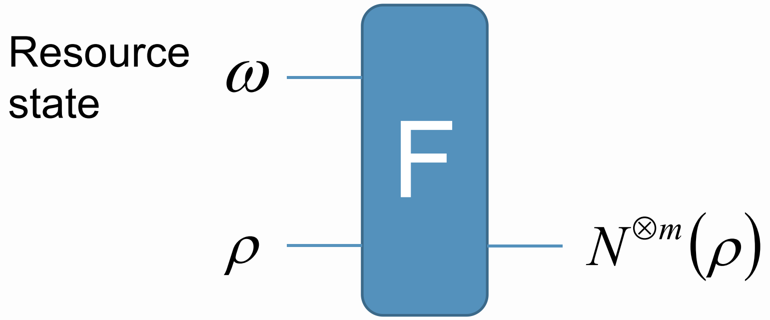
$$E_\kappa(\mathcal{N}_1 \otimes \mathcal{N}_2) = E_\kappa(\mathcal{N}_1) + E_\kappa(\mathcal{N}_2)$$

- We further have $E_{\mathrm{PPT}}(\mathcal{N}) \le \lim_{m\to\infty} \frac{1}{m}\log\left(2^{E_\kappa\left(\mathcal{N}^{\otimes m}\right)} + 2\right) \le \lim_{m\to\infty} \frac{1}{m}\log\left(2^{mE_\kappa(\mathcal{N})} + 2\right) = E_\kappa(\mathcal{N})$

Entanglement cost of a quantum channel

For a quantum channel $\mathcal{N}_{A\to B}$, the exact parallel entanglement cost of $\mathcal{N}_{A\to B}$ is equal to its κ-entanglement:

$$E_{\mathrm{PPT}}^{(p)}(\mathcal{N}_{A\to B}) = E_\kappa(\mathcal{N}_{A\to B}).$$

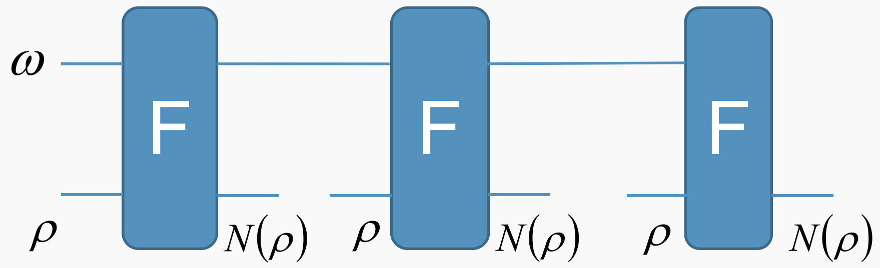


Sequential vs. Parallel channel simulation

Exact entanglement cost of sequential simulation

For any quantum channel $\mathcal{N}_{A\to B}$, the sequential exact entanglement cost is given by

$$E_{\mathrm{PPT}}(\mathcal{N}_{A\to B}) = E_\kappa(\mathcal{N}_{A\to B}).$$



- Our key contribution is the following sandwiched approximation

$$\log\left[2^{nE_\kappa(\mathcal{N})}-1\right] \le E_{\mathrm{PPT}}(\mathcal{N}_{A\to B}, n) \le \log\left[\frac{2^{(n+1)E_\kappa(\mathcal{N})}-1}{2^{E_\kappa(\mathcal{N})}-1}\right].$$

- Lower bound: sequential simulation cost $\geq$ parallel simulation cost
- Achievable part: A protocol that forces the resource after every round to be **maximally entangled** and **reuses** it.

Sequential and parallel protocols have the same power in this task of channel simulation!

Exact entanglement cost of quantum channel

As applications, we solve the (exact) entanglement cost for fundamental quantum channels including

(1) Erasure channel $\mathcal{E}_p(\rho) = (1-p)\rho + p|e\rangle\langle e|$:

$$E_C(\mathcal{E}^q) = E_C^{(p)}(\mathcal{E}^q) = (1-q)\log d,$$

$$E_{\mathrm{PPT}}{}^{(p)}(\mathcal{E}_p) = E_{\mathrm{PPT}}(\mathcal{E}_p) = \log(d(1-p)+p).$$

(2) Dephasing channels $\mathcal{D}_q(\rho) = (1-q)\rho + qZ\rho Z$:

$$E_C(\mathcal{D}^q) = E_C^{(p)}(\mathcal{D}^q) = h_2\left(\frac{1}{2} + \sqrt{q(1-q)}\right)$$

$$E_{\mathrm{PPT}}{}^{(p)}(\mathcal{D}^q) = E_{\mathrm{PPT}}(\mathcal{D}^q) = \log(1+2|q-1/2|).$$

(3) Depolarizing $\mathcal{N}_{D,p}(\rho) = (1-p)\rho + \frac{p}{d^2-1}\sum_{\substack{0\le i,j\le d-1\\(i,j)\neq(0,0)}} X^iZ^j\rho(X^iZ^j)^\dagger$:

$$E_{\mathrm{PPT}}(\mathcal{N}_{D,p}) = \begin{cases} \log d(1-p) & \text{if } 1-p \ge \frac{1}{d} \\ 0 & \text{if } 1-p < \frac{1}{d} \end{cases}$$

(4) Single-mode bosonic Gaussian channels: we give analytical solutions for E_C and E_{PPT}.

Thoughts on channel resource measures

- The exact entanglement cost of channel is equal to the maximum kappa entanglement generated by the quantum channel.

$$E_\kappa(\mathcal{N}_{A\to B}) = \sup_{\rho_{A'AB'}} [E_\kappa(\mathcal{N}_{A\to B}(\rho_{A'AB'})) - E_\kappa(\rho_{A'AB'})]$$

- Supports one way of introducing channel resource measures (amortized resourcefulness of a quantum channel, Kaur and Wilde'18)

$$E(\mathcal{N}_{A\to B}) = \sup_{\rho}[E(\mathcal{N}(\rho)) - E(\rho)]$$

- Our results on the exact entanglement cost of quantum channels are good examples of static resource cost of quantum channels under parallel and adaptive protocols.

- Similar ideas work for the resource theory of coherence, Díaz et al.18 showed the one-shot coherence simulation cost under MIO is characterized by the max-channel divergence

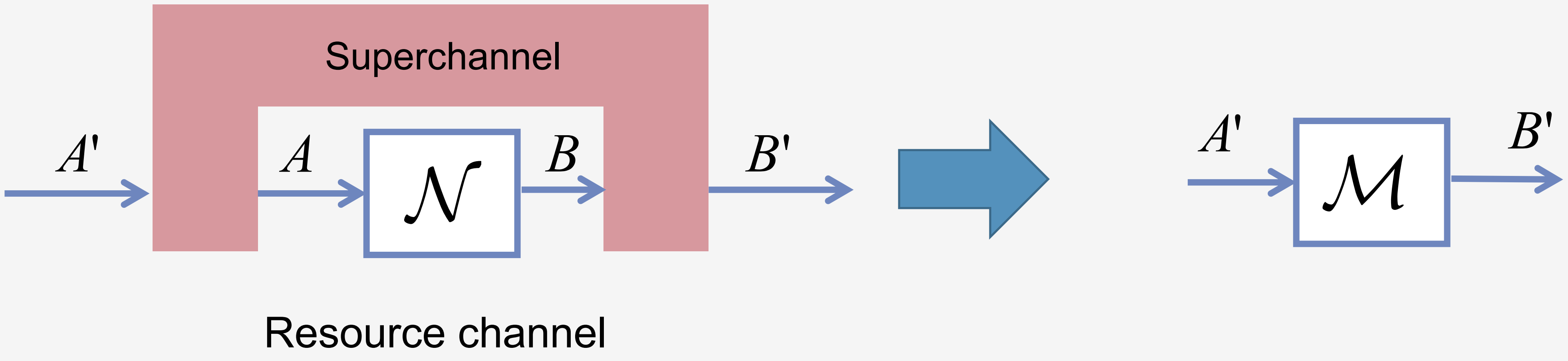
$$S_{c,\mathrm{MIO}}^{(1),\varepsilon}(\mathcal{N}) = \min_{\mathcal{M}\in\mathrm{MIO}} D_{\max}^{\varepsilon}(\mathcal{N}\|\mathcal{M})$$

- Supports the resource measures via channel divergences (Cooney, Mosonyi, Wilde'16; Leditzky et al.'18)

$$\mathbf{D}(\mathcal{N}\|\mathcal{M}) := \max_{|\varphi\rangle_{RA}} \mathbf{D}\left(\mathcal{N}_{A\to B}(\varphi_{RA}) \| \mathcal{M}_{A\to B}(\varphi_{RA})\right)$$

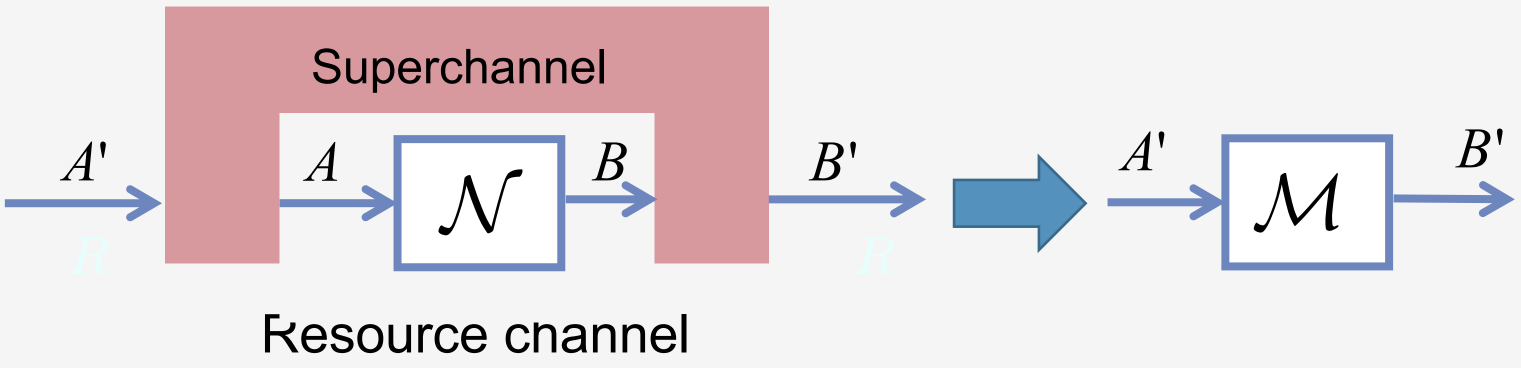
Dynamic resource cost of quantum channels

Dynamic resource cost of quantum channels



- In general, there are free quantum channels and free superchannels (bipartite quantum channels that sends channels to channels, even when tensored with the identity map).
- What are the minimal dynamic resources that are required to realize another quantum channel?
- Toy example - quantum teleportation
  - Recall that 2 cbits + 1 ebit → 1 qbit
  - When shared entanglement is free, we need a two-bit classical noiseless channel to simulate a noiseless qubit channel.

Dynamic resource cost of quantum channels



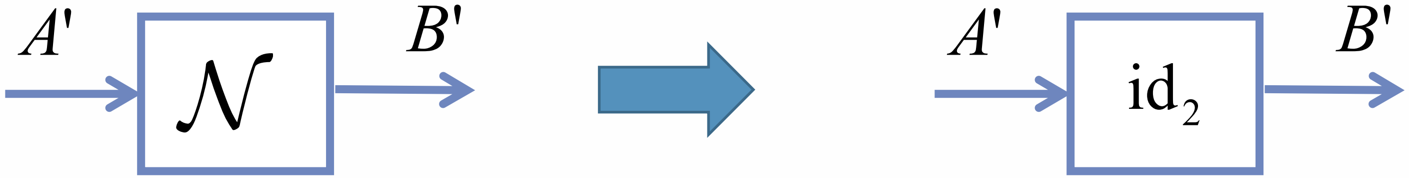
The minimum error of simulation from N to M with Ω free operations is defined as

$$\omega_\Omega(\mathcal{N},\mathcal{M}) := \frac{1}{2}\inf_{\Pi\in\Omega} \| \Pi \circ \mathcal{N} - \mathcal{M}\|_\diamond$$

The channel simulation rate from N to M is then defined as

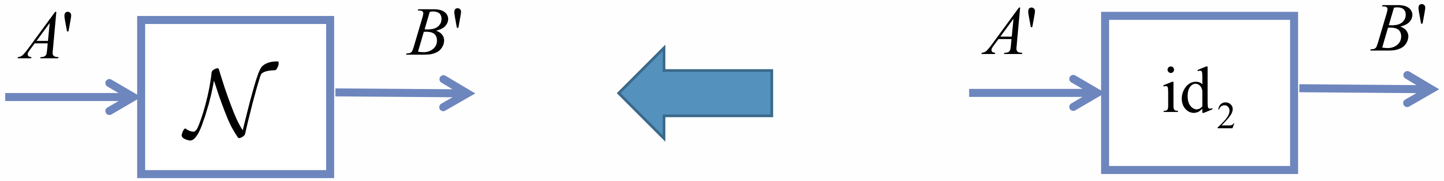
$$S_\Omega(\mathcal{N},\mathcal{M}) := \lim_{\varepsilon\to 0} \inf \left\{ \frac{n}{m} : \omega_\Omega\left(\mathcal{N}^{\otimes n}, \mathcal{M}^{\otimes m}\right) \le \varepsilon \right\}$$

Dynamic resource transformation - channel capacity vs simulation cost



- If we want to use noisy channels to simulate noiseless channels, the cost is indeed relates to the quantum capacity.
- Optimal rate to simulate the identity channel via channel N?

$$Q_{\Omega}(\mathcal{N}) = S_{\Omega}\left(\mathcal{N}, \mathrm{id}_2\right)^{-1}$$



- Instead, what is the optimal rate to simulate a channel N via the identity channel?

$$S_{\Omega}(\mathcal{N}) := S_{\Omega}\left(\mathrm{id}_2, \mathcal{N}\right)$$

- By operational reasons, we have $Q_{\mathrm{E}}(\mathcal{N}) \le Q_{\mathrm{NS}}(\mathcal{N}) \le S_{\mathrm{NS}}(\mathcal{N}) \le S_{\mathrm{E}}(\mathcal{N})$

Max-information of a quantum channel (FWTB, 1807.05354)

Resource theory perspectives

- Free dynamic resources F: constant channels
- Free superchannels: LO with entanglement/NS correlations
- Motivate us to define the following resource measure

$$D_{\max}(\mathcal{N} \| \mathcal{M}) := D_{\max}\left(J_{\mathcal{N}} \| J_{\mathcal{M}}\right)$$

$$I_{\max}(A:B)_{\mathcal{N}} = \inf_{\mathcal{M}\in\mathcal{F}} D_{\max}(\mathcal{N} \| \mathcal{M})$$

New tools

The channel's smooth max-information is defined by

$$I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}} := \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}-\mathcal{N}\|_{\diamond}\leq\varepsilon} I_{\max}(A:B)_{\widetilde{\mathcal{N}}}$$

The above is also compatible with other channel divergence.

Asymptotic simulation cost

Main results

For any quantum channel $\mathcal{N}_{A'\to B}$ and given error tolerance $\varepsilon \geq 0$, we show

$$S^{(1)}_{\mathrm{NS},\varepsilon}(\mathcal{N}) = \frac{1}{2} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}}.$$

The channel's smooth max-information has the asymptotic equipartition property,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} = I(A:B)_{\mathcal{N}}.$$

Techniques

The direct proof of this AEP involves

- Post-selection technique (Christandl, Koenig, Renner'08; Berta, Christandl, Renner'11)
- Partial smooth bound (Anshu, Berta, Jain, Tomamichel'18)
- AEP for states (Tomamichel, Colbeck, Renner'08)

Discussions

- How to better characterize the one-shot entanglement-assisted channel simulation cost?
- What is the simulation cost under the adaptive or sequential regime?

Another Example of dynamic cost: gate synthesis

- Magic states and channels are necessary resources to achieve universal QC.
- QC via magic state manipulation is one popular model for realizing FTQC.
- Quantify the magic for quantum channels (XW, Wilde, Su'19), see also (Seddon, Campbell'19 for multi-qubit operations).
- Resource measures (e.g., channel divergences) help us estimate the magic cost of (noisy) quantum circuits.

For any qudit quantum channel $\mathcal{N}$, the number of channels $\mathcal{E}$ required to implement it is bounded from below as follows:

$$S_{\mathcal{E}}(\mathcal{N}) \geq \max\left\{\frac{\mathcal{M}(\mathcal{N})}{\mathcal{M}(\mathcal{E})}, \frac{\theta_{\max}(\mathcal{N})}{\theta_{\max}(\mathcal{E})}\right\}.$$

Application of resource theory to quantum channel discrimination

(Wang, Wilde,1907.06306)

Bit of asymmetric distinguishability

- Motivation: Distinguishability is a resource in the sense that it limits the amount of effort needed to make decisions.
- Distinguishability is a resource that can be quantified and interconverted.

- We introduce the fundamental unit called bit of asymmetric distinguishability (AD):

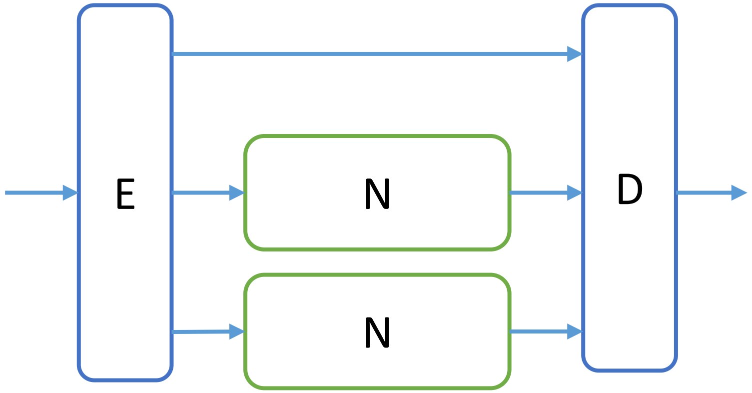
$$(|0\rangle\langle 0|, \pi) \text{ with } \pi = I/2.$$

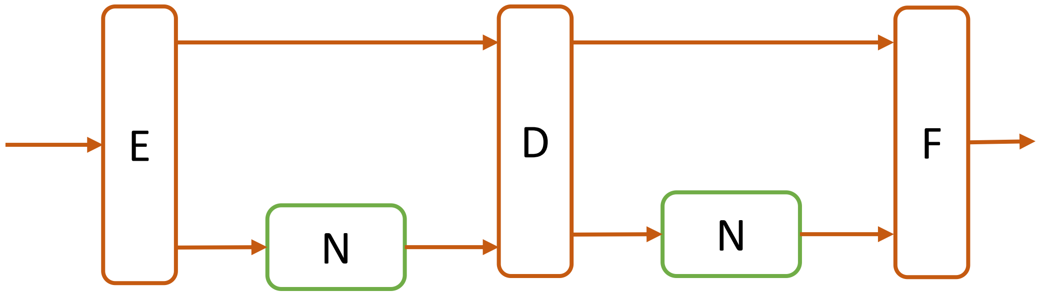
- In general, $logM$ bits of asymmetric distinguishability are encoded in the following qubit state box:

$$(|0\rangle\langle 0|, \pi_M) \quad \text{with } \pi_M = 1/M\, |0\rangle\langle 0| + (1 - 1/M)|1\rangle\langle 1|.$$

Resource theory of AD for channels

- The basic object to manipulate is a channel box, consisting of two channels.
- Free operations: **superchannel** (Chiribella et al.'08), which accepts as input a quantum channel and outputs a channel, e.g., $\Pi_{(A\to B)\to(C\to D)}(\mathcal{N}_{A\to B}) = \mathcal{K}_{C\to D}$.
- Distinguishability of channels is fundamental and resourceful.
- Tasks: distillation, dilution, channel box transformations.
- Currency boxes: $(\mathcal{R}^{|0\rangle\langle 0|}, \mathcal{R}^{\pi_M})$
- Important point: parallel strategy and sequential strategy.





Asymptotic rates

Asymptotic parallel distillable distinguishability is given by

$$D_d^p(\mathcal{N},\mathcal{M}) := \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}D_d^{\varepsilon}\left(\mathcal{N}^{\otimes n},\mathcal{M}^{\otimes n}\right) = \lim_{m\to\infty}\frac{1}{m}D(\mathcal{N}\|\mathcal{M})$$

where $D(\mathcal{N}\|\mathcal{M}) := \sup_{\psi_{RA}} D\left(\mathcal{N}_{A\to B}(\psi_{RA})\|\mathcal{M}_{A\to B}(\psi_{RA})\right)$ is the channel relative entropy.

Asymptotic sequential distillable distinguishability = amortized channel relative entropy:

$$D_d(\mathcal{N},\mathcal{M}) := \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}D_d^{\varepsilon}\left(\mathcal{N}^{(n)},\mathcal{M}^{(n)}\right) = D_{\mathcal{A}}(\mathcal{N}\|\mathcal{M}),\ \textit{with}$$

$$D_{\mathcal{A}}(\mathcal{N}\|\mathcal{M}) := \sup_{\rho_{RA},\sigma_{RA}} D\left(\mathcal{N}_{A\to B}(\rho_{RA})\|\mathcal{M}_{A\to B}(\sigma_{RA})\right) - D(\rho_{RA}\|\sigma_{RA})$$

- **Operational meaning** for amortized channel relative entropy (Berta, Hirche, Kaur, Wilde'18).
- A solution to Stein's lemma for quantum channels in the **sequential** setting.

Parallel vs. sequential

- A natural question is whether sequential distillable distinguishability can be larger than the parallel distillable distinguishability.

Parallel protocol $D_{\text{reg}}(\mathcal{N}\|\mathcal{M})$ **vs.** Sequential protocol $D_{\mathcal{A}}(\mathcal{N}\|\mathcal{M})$

- Because non-adaptive strategies are a special case of adaptive strategies

$$D_{\text{reg}}(\mathcal{N}\|\mathcal{M}) \leq D_{\mathcal{A}}(\mathcal{N}\|\mathcal{M})$$

- Fang, Fawzi, Renner, Sutter'19 introduced a chain rule of relative entropy, which leads to a negative answer of the above question!

$$D(\mathcal{N}(\rho_{RA})\|\mathcal{M}(\sigma_{RA})) \leq D(\rho_{RA}\|\sigma_{RA}) + D_{\text{reg}}(\mathcal{N}\|\mathcal{M})$$

$$\Rightarrow D_{\text{reg}}(\mathcal{N}\|\mathcal{M}) = D_{\mathcal{A}}(\mathcal{N}\|\mathcal{M})$$

Summary

- Quantum resource theory of quantum channels provides us a powerful framework to understand the fundametal quantum resource cost and power of quantum channels.
- operational and quantitative insights into various quantum information processing tasks
- More fruitful structure:

Manipulation type	Cost	Generation
Regime or strategy	Parallel	Sequential
Resource type	Static	Dynamic

- Motivate different types of resource measures for quantum channels (amortized measure and channel divergence measure);
- Applications to many fields, e.g., channel discrimination.

Outlook

- Quantum reverse Shannon theorem under the sequential regime? What is the sequential entanglement-assisted simulation cost of quantum channels?
- Entanglement cost of quantum channels under the sequential regime?
- Tools for static-resource and dynamic-resource sequential regime.
- Further development of the theory of the channel divergences.
- Parallel versus Sequential strategies?
    - Does the sequential strategy have advantages over the parallel strategy in some quantum resource theory of channels?

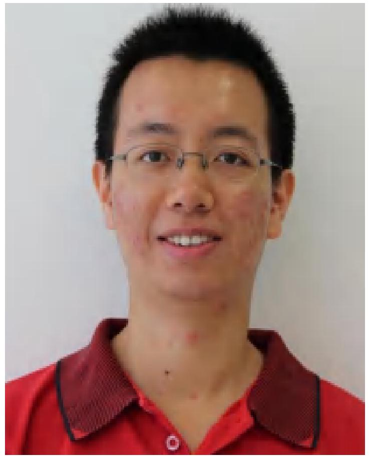
Thanks for your attention!

Slides available at www.xinwang.info

I would like to thank my collaborators

M. Berta

K. Fang



Y. Su

M. Tomamichel

M. M. Wilde