Optimizing the fundamental limits for quantum communication

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Quantum capacity of a quantum channel

- The quantum capacity of a channel $N$ is the number of qubits, on average, that can be faithfully transmitted with each use of $N$.

- The task is to protect quantum information from errors due to quantum noise (or simulate a noiseless quantum channel).

- Quantum capacity theorem is established by (Lloyd, Shor, Devetak 97-05) & (Barnum, Nielsen, Schumacher 96-00)

$$Q(N) = \lim_{m \to \infty} \frac{1}{m} I_c\left( N^\otimes m \right)$$

- Coherent information

$$I_c(N) := \max_\rho \left[ H(N(\rho)) - H(\mathcal{N}^c(\rho)) \right]$$
Difficulty of estimating $Q(N)$

- $Q(N)$ does not have a single-letter formula.
- Regularization is necessary in general [Cubitt el.al, 2014].
- $Q(N)$ is not additive in general [Smith, Yard, 2009].

- Even for qubit depolarizing channel

$$D_p(\rho) := (1 - p)\rho + p \text{Tr}(\rho)I / 2$$

we do not know its quantum capacity.
Many methods to estimate $Q(N)$

- To evaluate $Q(N)$, substantial efforts have been made in the past two decades.
- Bounds for general channels
  - Partial transposition bound (Holevo, Werner'01)
  - Rains information (Tomamichel, Wilde, Winter'14)
  - max-Rains bound (Wang, Fang, Duan'18)
  - Geometric Rényi bound (Fang, Fawzi'19)
  - ...
- Bounds for depolarizing channel
  - Symmetric Side Channels (Smith, Smolin, Winter'08)
  - Approximate degradable channels (Sutter et al.'17)
  - Degradable decomposition (Leditzky, Datta, Smith'18)
  - Quantum flag bound (Fanizza, Kianvash, Giovannetti'19)
Main messages

• New single-letter fundamental limits for entanglement distillation, quantum communication, and private communication.

• Optimize the extended channel whose quantum capacity is easy to estimate.

• Improved bounds for several fundamental quantum channels.
Main result 1 - Bound for distillable entanglement

\[ \rho_{AB} \otimes^n \xrightarrow{\Phi^{\otimes m}} \text{weaker than} \rho_{ABF} \text{ with } \rho_{AB} = Tr_F \rho_{ABF} \]

- Apply the converse bound via approximate degradability bound Leditzky et al. 2017
  \[ E_{D,\rightarrow}(\rho_{AB}) \leq E_{D,\rightarrow}(\rho_{ABF}) \leq I(A\rangle BF)\rho + f(\eta(\rho)_{A\mid BF}) \]
  where \( \eta(\rho) \) is degradability parameter.

- How can we further optimize over the extended states?
  **Parametrize the extended state!**

- Consider the sub-state decomposition
  \[ \rho_{AB} = \tau_{AB} + \omega_{AB} \]

  \[ \rho_{ABF} = \tau_{AB} \otimes |\psi_\alpha\rangle\langle\psi_\alpha| + \omega_{AB} \otimes |0\rangle\langle0| \]

  \[ E_{D,\rightarrow}(\rho_{AB}) \leq \inf_{0 \leq \alpha \leq 1} s(\tau_{AB}, \omega_{AB}, \alpha) \]
  function \( s \) can be efficiently computed.
Main result 2 - new upper bound for depolarizing channel

- One of the most important channels, useful in modelling noise for quantum hardware.

\[ D_p(\rho) := (1 - p)\rho + p \text{Tr}(\rho)I / 2 \]

- However, its quantum capacity remains unsolved despite substantial efforts.

- \( Q(N) \) of a teleportation-simulable \( N = \text{one-way distillable entanglement of its Choi state} \).
  (Bennett et al'96)
- The Choi state of the qubit depolarizing channel \( \rho_{D_p} = (1 - p)\Phi + pI / 4 \)

- Applying our bound on one-way distillable entanglement

\[ Q(D_p) \leq \inf_{0 \leq \alpha \leq 1} s((1 - p)\Phi, pI / 4, \alpha) \]

- The final step is to search over \( \alpha \).
• We establish **improved upper bounds** on $Q(N)$ of the depolarizing channel.
Upper bounds via flags and degradability

For a general quantum channel, we could deploy the quantum flags (Fanizza et al.'19).

For a channel with CP map decomposition,  
\[ \mathcal{N} = \sum_{j=0}^{k} \mathcal{N}_j \]

\[ Q(\mathcal{N}) \leq \inf_{\sigma_0, \ldots, \sigma_k} Q^{(1)}(\widehat{\mathcal{N}}) + f(\eta(\widehat{\mathcal{N}})) \]

\[ \text{with} \quad \widehat{\mathcal{N}}(\cdot) = \sum_{j=0}^{k} \mathcal{N}_j(\cdot) \otimes \sigma_j \]

We could take a more specific structure  
\[ \widehat{\mathcal{N}}(\cdot) = \mathcal{N}_0(\cdot) \otimes |\psi_\alpha\rangle\langle\psi_\alpha| + \mathcal{N}_1(\cdot) \otimes |0\rangle\langle0| \]

\[ Q(\mathcal{N}) \leq P(\mathcal{N}) \leq \widetilde{Q}_{pf}(\mathcal{N}) := \inf_{0 \leq \alpha \leq 1} \left\{ Q^{(1)}(\widehat{\mathcal{N}}) |\eta(\widehat{\mathcal{N}})| = 0 \right\} \]

Q1 of degradable extended channels can be efficiently computed (Fawzi & Fawzi'17)

Difference between our method and the quantum flag method in Fanizza et al.'19

1. We parametrize flags and then optimize over them!
2. We consider a general CP map decomposition.
Application 2 - Generalized amplitude damping channel

- The GAD channel is one of the realistic sources of noise in practice.

\[ \mathcal{A}_{y,N}(\rho) = A_1 \rho A_1^\dagger + A_2 \rho A_2^\dagger + A_3 \rho A_3^\dagger + A_4 \rho A_4^\dagger \]

\[ A_1 = \sqrt{1-N} |0\rangle\langle 0| + \sqrt{1-y} |1\rangle\langle 1| \quad A_2 = \sqrt{y(1-N)} |0\rangle\langle 1| \quad A_3 = \sqrt{N(1-y)} |0\rangle\langle 0| + |1\rangle\langle 1| \quad A_4 = \sqrt{yN} |1\rangle\langle 0| \]

- We introduce the extended channel

\[ \hat{\mathcal{A}}_{y,N} = \sqrt{1-N} \mathcal{A}_{y,N}^1(\rho) \otimes |\psi_\alpha\rangle\langle \psi_\alpha| + \sqrt{N} \mathcal{A}_{y,N}^2(\rho) \otimes |0\rangle\langle 0| \]

- By numerics and analysis, we find that \( \alpha = 0 \) would be the best choice.
- We further show that

\[ Q(\mathcal{A}_{y,N}) \leq Q\left(\hat{\mathcal{A}}_{y,N}\right) = \max_{p \in [0,1]} I_c \left( p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|, \hat{\mathcal{A}}_{y,N} \right) \]
Our bound is tighter than previous upper bounds in (Khatri et al.'19) via the data processing approach (Khatri et al.'19) and Rains information (Tomamichel'14).
Application 3 - BB84 channel

Independent bit and phase error

\[ B_{p_x,p_z}(\rho) = (1 - p_X - p_Z + p_Xp_Z)\rho + (p_X - p_Xp_Z)X\rho X + (p_Z - p_Zp_X)Z\rho Z + p_Xp_Z Y\rho Y \]

\[ Q^{(1)}(B_{p_x,p_z}) = 1 - h(p_X)0 - h(p_Z) \]

Smith and Smolin '08

\[ Q(B_{p,p}) \leq h\left(\frac{1}{2} - 2p(1-p)\right) - h(2p(1-p)) \]

Sutter et al.'2017 improved the bound in the region \(0 < p < 0.0002\)

We have established improved bounds for this channel.
Summary

- The key idea is to optimize the extended channels.
- The extended or flagged channel structure is quite useful and can be combined with other techniques of channel capacity estimation.
- Improved upper bounds on the quantum/private capacities of depolarizing channel, BB84 channel, generalized amplitude damping channel.

Outlook

- It will be interesting to look at the interaction between extended channels and the degradable and anti-degradable decomposition of channels (Leditzky et al.'18).
- Apply the techniques in this work to classical capacity or other resource theories.
Thanks for your attention!


Slides available at www.xinwang.info